A description of the quantum superalgebra $U_q\,[sl(n+1|m)]$ via creation and annihilation generators

1999 J. Phys. A: Math. Gen. 32 1053

(http://iopscience.iop.org/0305-4470/32/6/019)

J. Phys. A: Math. Gen. **32** (1999) 1053–1064. Printed in the UK PII: S0305-4470(99)98391-9

A description of the quantum superalgebra $U_q[sl(n+1|m)]$ via creation and annihilation generators

T D Palev† and N I Stoilova†

Abdus Salam International Centre for Theoretical Physics, 34100 Trieste, Italy

Received 13 October 1998

Abstract. A description of the quantum superalgebra $U_q[sl(n+1|m)]$ and in particular of the special linear superalgebra $sl(n+1|m)$ via creation and annihilation generators (CAGs) is given. It provides an alternative to the canonical description of $U_q[sl(n+1|m)]$ in terms of Chevalley generators. A conjecture that the Fock representations of the CAGs provide microscopic realizations of exclusion statistics is formulated.

1. Introduction

The description of the quantized simple (universal enveloping) Lie algebras [1, 2] and the basic Lie superalgebras [3–7] is usually carried out in terms of their Chevalley generators ($e_i, f_i, h_i, i = 1, \ldots, n$, for an algebra of rank n). Recently it has been pointed out that the quantum (super)algebras $U_q[osp(1|2n)]$ [8–10], $U_q[so(2n+1)]$ [11], more generally $U_q[osp(2r+1|2m)]$, $r+m=n$ [12], and also $U_q[sl(n+1)]$ [13] can be defined via alternative sets of generators $a_i^{\pm}, H_i, i = 1, \ldots, n$, referred to as (deformed) creation and annihilation generators (CAGs) or creation and annihilation operators.

The concept of creation and annihilation generators of a simple Lie (super)algebra was introduced in [14]. Let $\mathcal{A}$ be such an algebra with a supercommutator $[\![,]\!]$. The root vectors $a_1^{\xi}, \ldots, a_n^{\xi}$ of $\mathcal{A}$ are said to be creation ($\xi = +$) and annihilation ($\xi = -$) generators of $\mathcal{A}$, if

$$\mathcal{A} = \text{lin.env.}\{a_i^{\xi}, [\![a_j^{\eta}, a_k^{\varepsilon}]\!] | i, j, k = 1, \ldots, n; \xi, \eta, \varepsilon = \pm\} \quad (1)$$

so that $a_1^+, \ldots, a_n^+$ (resp. $a_1^-, \ldots, a_n^-$) are negative (resp. positive) root vectors of $\mathcal{A}$.

The justification for such terminology stems from the observation that the creation and the annihilation generators of the orthosymplectic Lie superalgebra (LS) $osp(2r+1|2m)$ have a direct physical significance: $a_1^{\pm}, \ldots, a_m^{\pm}$ (resp. $a_{m+1}^{\pm}, \ldots, a_n^{\pm}$) are para-Bose (resp. para-Fermi) operators [15], namely operators which generalize the statistics of the tensor (resp. spinor) fields in quantum field theory [16]. The LS $osp(2r+1|2m)$ is an algebra from the class B in the classification of Kac [17]. Therefore the paraquantizations (and hence the canonical Bose and Fermi quantization) could be called B-quantizations (or, more precisely, representations of a B-quantization).

A conjecture, stated in [18], assumes that to each class A, B, C and D of basic LSs [17] there corresponds a quantum statistics, so that its CAGs can be interpreted as creation and annihilation operators of real particles in the corresponding Fock space(s). This conjecture

† Permanent address: Institute for Nuclear Research and Nuclear Energy, 1784 Sofia, Bulgaria; E-mail addresses: tpalev@inrne.bas.bg and stoilova@inrne.bas.bg

holds for the classes A, B, C and D of simple Lie algebras [19]. It was studied in more detail for the Lie algebras $sl(n+1)$ (A-statistics) [20] and for the LSs $sl(1|m)$ (A-superstatistics) [14, 21]. As an illustration we mention that the Wigner quantum systems (WQSs), introduced in [22], are based on the A-superstatistics. These systems, which attracted some attention from different points of view [23–25], possess quite unconventional physical properties. For example, the $(n+1)$-particle WQS, based on the LS $sl(1|3n)$ [26], exhibits a quark-like structure: the composite system occupies a small volume around the centre of mass and within it the geometry is noncommutative. The underlying statistics is a Haldane exclusion statistics [27], a subject of considerable interest in condensed matter physics.

We are not going to discuss further the properties of the superstatistics (for more details on this subject see [28, 26] and the references therein). We mention this point here only in order to indicate that the alternative description of $sl(n+1|m)$ and $U_q[sl(n+1|m)]$ will be carried out in terms of (deformed) creation and annihilation generators, which, contrary to the Chevalley generators, could also be of direct physical relevance.

Throughout the paper we use the notation: LS, LS's—Lie superalgebra, Lie superalgebras; CAGs—creation and annihilation generators; lin.env.—linear envelope; $\mathbb{Z}$—all integers; $\mathbb{Z}_+$—all non-negative integers; $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$—the ring of all integers modulo 2; $\mathbb{C}$—all complex numbers;

$$[p; q] = \{p, p+1, p+2, \ldots, q-1, q\} \qquad \text{for} \quad p \leqslant q \in \mathbb{Z} \tag{2}$$

$$\theta_i = \begin{cases} \bar{0} & \text{if } i = 0, 1, 2, \ldots, n \\ \bar{1} & \text{if } i = n+1, n+2, \ldots, n+m \end{cases} \qquad \theta_{ij} = \theta_i + \theta_j \tag{3}$$

$$[a, b] = ab - ba \quad \{a, b\} = ab + ba \quad [\![a, b]\!] = ab - (-1)^{\deg(a)\deg(b)} ba \tag{4}$$

$$[a, b]_x = ab - xba \quad \{a, b\}_x = ab + xba \quad [\![a, b]\!]_x = ab - (-1)^{\deg(a)\deg(b)} xba. \tag{5}$$

2. The Lie superalgebra $sl(n+1|m)$

Here we give an alternative definition of the special linear Lie superalgebra $sl(n+1|m)$ in terms of creation and annihilation generators $a_1^\pm, a_2^\pm, \ldots, a_{n+m}^\pm$. We write down the relations between the CAGs and the Chevalley generators.

First we recall that the universal enveloping algebra $U[gl(n+1|m)]$ of the general linear LS $gl(n+1|m)$ is a $\mathbb{Z}_2$-graded associative unital superalgebra generated by $(n+m+1)^2$ $\mathbb{Z}_2$-graded indeterminates $\{e_{ij} | i, j \in [0; n+m]\}$, $\deg(e_{ij}) = \theta_{ij}$, subject to the relations

$$[\![e_{ij}, e_{kl}]\!] = \delta_{jk} e_{il} - (-1)^{\theta_{ij}\theta_{kl}} \delta_{il} e_{kj} \qquad i, j, k, l \in [0; n+m]. \tag{6}$$

The LS $gl(n+1|m)$ is a subalgebra of $U[gl(n+1|m)]$, considered as a Lie superalgebra, with generators $\{e_{ij} | i, j \in [0; n+m]\}$ and supercommutation relations (6). The LS $sl(n+1|m)$ is a subalgebra of $gl(n+1|m)$:

$$sl(n+1|m) = \text{lin.env.}\{e_{ij}, (-1)^{\theta_k} e_{kk} - (-1)^{\theta_l} e_{ll} | i \neq j; i, j, k, l \in [0; n+m]\}. \tag{7}$$

The generators $e_{00}, e_{11}, \ldots, e_{n+m,n+m}$ constitute a basis in the Cartan subalgebra of $gl(n+1|m)$. Denoted by $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n+m}$ the dual basis, $\varepsilon_i(e_{jj}) = \delta_{ij}$. The root vectors of both $gl(n+1|m)$ and $sl(n+1|m)$ are e_{ij}, $i \neq j$, $i, j \in [0; n+m]$. The root corresponding to e_{ij} is $\varepsilon_i - \varepsilon_j$. With respect to the natural order of the basis in the Cartan subalgebra e_{ij} is a positive (resp. a negative) root vector if $i < j$ (resp. $i > j$).

The above description of $sl(n+1|m)$ is simple, but it is not appropriate for quantum deformations. A more 'economic' definition is given in terms of the Chevalley generators

$$\hat{h}_i = e_{i-1,i-1} - (-1)^{\theta_{i-1,i}} e_{ii} \quad \hat{e}_i = e_{i-1,i} \quad \hat{f}_i = e_{i,i-1} \qquad i \in [1; n+m] \tag{8}$$

and the $(n+m)\times(n+m)$ Cartan matrix $\{\alpha_{ij}\}$ with entries

$$\alpha_{ij} = (1+(-1)^{\theta_{i-1,i}})\delta_{ij} - (-1)^{\theta_{i-1,i}}\delta_{i,j-1} - \delta_{i-1,j} \qquad i, j \in [1; n+m]. \tag{9}$$

We are working with a nonsymmetric Cartan matrix [17]. For instance the Cartan matrix (9), corresponding to $n+1=3$, $m=5$ is 7×7 dimensional matrix:

$$(\alpha_{ij}) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{10}$$

$U[sl(n+1|m)]$ is an associative unital algebra of the Chevalley generators, subject to the Cartan–Kac relations

$$[\hat{h}_i, \hat{h}_j] = 0 \quad [\hat{h}_i, \hat{e}_j] = \alpha_{ij}\hat{e}_j \quad [\hat{h}_i, \hat{f}_j] = -\alpha_{ij}\hat{f}_j \quad [\![\hat{e}_i, \hat{f}_j]\!] = \delta_{ij}\hat{h}_i \tag{11}$$

and the Serre relations

$$[\hat{e}_i, \hat{e}_j] = 0 \quad [\hat{f}_i, \hat{f}_j] = 0 \quad \text{if} \quad |i-j| \neq 1 \tag{12a}$$

$$\hat{e}_{n+1}^2 = 0 \quad \hat{f}_{n+1}^2 = 0 \tag{12b}$$

$$[\hat{e}_i, [\hat{e}_i, \hat{e}_{i+1}]] = 0 \quad [\hat{f}_i, [\hat{f}_i, \hat{f}_{i+1}]] = 0 \qquad i \neq n+m \tag{12c}$$

$$[\hat{e}_{i+1}, [\hat{e}_{i+1}, \hat{e}_i]] = 0 \quad [\hat{f}_{i+1}, [\hat{f}_{i+1}, \hat{f}_i]] = 0 \qquad i \neq n+m \tag{12d}$$

$$\{[\hat{e}_{n+1}, \hat{e}_n], [\hat{e}_{n+1}, \hat{e}_{n+2}]\} = 0 \qquad \{[\hat{f}_{n+1}, \hat{f}_n], [\hat{f}_{n+1}, \hat{f}_{n+2}]\} = 0. \tag{12e}$$

The so-called additional Serre relations (12*e*) [29–31] can also be written in the form

$$\{\hat{e}_{n+1}, [[\hat{e}_n, \hat{e}_{n+1}], \hat{e}_{n+2}]\} = 0 \qquad \{\hat{f}_{n+1}, [[\hat{f}_n, \hat{f}_{n+1}], \hat{f}_{n+2}]\} = 0. \tag{12f}$$

The grading on $U[sl(n+1|m)]$ is induced from the requirement that the only odd generators are $\hat{e}_{n+1}$ and $\hat{f}_{n+1}$, namely

$$\deg(\hat{h}_i) = \hat{0} \qquad \deg(\hat{e}_i) = \deg(\hat{f}_i) = \theta_{i-1,i}. \tag{13}$$

The LS $sl(n+1|m)$ is a subalgebra of $U[sl(n+1|m)]$, generated by the Chevalley generators in a sense of a Lie superalgebra. It is a linear span of the Chevalley generators (8) and all root vectors

$$\begin{aligned} e_{ij} &= [[[\ldots[[\hat{e}_{i+1}, \hat{e}_{i+2}], \hat{e}_{i+3}], \ldots], \hat{e}_{j-1}], \hat{e}_j] \\ e_{ji} &= [\hat{f}_j, [\hat{f}_{j-1}, [\ldots, [\hat{f}_{i+2}, \hat{f}_{i+1}]\ldots,]]] \qquad i+1<j \quad i, j \in [0; n+m]. \end{aligned} \tag{14}$$

Consider the following root vectors from $sl(n+1|m)$:

$$\hat{a}_i^+ = e_{i0} \qquad \hat{a}_i^- = e_{0i} \qquad i \in [1; n+m] \tag{15}$$

or, equivalently from (14)

$$\hat{a}_1^- = \hat{e}_1 \quad \hat{a}_i^- = [[[\ldots[[\hat{e}_1, \hat{e}_2], \hat{e}_3], \ldots], \hat{e}_{i-1}], \hat{e}_i] = [\hat{a}_{i-1}^-, \hat{e}_i] \qquad i \in [2; n+m] \tag{16a}$$

$$\hat{a}_1^+ = \hat{f}_1 \quad \hat{a}_i^+ = [\hat{f}_i, [\hat{f}_{i-1}, [\ldots, [\hat{f}_3, [\hat{f}_2, \hat{f}_1]]\ldots]]] = [\hat{f}_i, \hat{a}_{i-1}^+] \qquad i \in [2; n+m]. \tag{16b}$$

The root of a_i^- (resp. of a_i^+) is $\varepsilon_0 - \varepsilon_i$ (resp. $\varepsilon_i - \varepsilon_0$). Therefore, (with respect to the natural order of the basis $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n+m}$) $a_1^-, \ldots, a_{n+m}^-$ are positive root vectors, whereas $a_1^+, \ldots, a_{n+m}^+$ are negative root vectors. Moreover, equation (1) with $\mathcal{A} = sl(n+1|m)$ holds. Hence, the generators (15) are creation and annihilation generators of $sl(n+1|m)$. These generators satisfy the following triple relations:

$$[\![\hat{a}_i^\xi, \hat{a}_j^\xi]\!] = 0 \qquad \xi = \pm \qquad i, j = 1, 2, \ldots, n+m \tag{17a}$$

$$[\![[\![\hat{a}_i^+, \hat{a}_j^-]\!], \hat{a}_k^+]\!] = \delta_{jk}\hat{a}_i^+ + (-1)^{\theta_i}\delta_{ij}\hat{a}_k^+ \qquad i, j, k = 1, 2, \ldots, n+m \tag{17b}$$

$$[\![[\![\hat{a}_i^+, \hat{a}_j^-]\!], \hat{a}_k^-]\!] = -(-1)^{\theta_{ij}\theta_k}\delta_{ik}\hat{a}_j^- - (-1)^{\theta_i}\delta_{ij}\hat{a}_k^- \qquad i, j, k \in [1; n+m]. \tag{17c}$$

The CAGs (15) together with (17) define completely $sl(n+1|m)$. The relations (17) are, however, (similar as equations (6)) not convenient for quantization. It turns out, and this is a new result, that one can take only a part of the relations (17), so that they still define completely $sl(n+1|m)$ and, as we shall see, are appropriate for Hopf algebra deformations.

Proposition 1. *$U[sl(n+1|m)]$ is an associative unital superalgebra with generators $\hat{a}_i^\pm$, $i \in [1; n+m]$ and relations:*

$$[\![\hat{a}_1^\xi, \hat{a}_2^\xi]\!] = 0 \qquad [\![\hat{a}_1^\xi, \hat{a}_1^\xi]\!] = 0 \qquad \xi = \pm \tag{18a}$$

$$[\![[\![\hat{a}_i^+, \hat{a}_j^-]\!], \hat{a}_k^+]\!] = \delta_{jk}\hat{a}_i^+ + (-1)^{\theta_i}\delta_{ij}\hat{a}_k^+ \quad |i-j| \leqslant 1 \qquad i, j, k \in [1; n+m] \tag{18b}$$

$$[\![[\![\hat{a}_i^+, \hat{a}_j^-]\!], \hat{a}_k^-]\!] = -(-1)^{\theta_{ij}\theta_k}\delta_{ik}\hat{a}_j^- - (-1)^{\theta_i}\delta_{ij}\hat{a}_k^- \quad |i-j| \leqslant 1 \qquad i, j, k \in [1; n+m]. \tag{18c}$$

The $\mathbb{Z}_2$-grading in $U[sl(n+1|m)]$ is induced from

$$\deg(\hat{a}_i^\pm) = \theta_i. \tag{19}$$

The proof follows from the expressions of the Chevalley generators (8) via the CAGs:

$$\hat{h}_1 = [\![\hat{a}_1^-, \hat{a}_1^+]\!] \qquad \hat{h}_i = (-1)^{\theta_{i-1}}([\![\hat{a}_i^-, \hat{a}_i^+]\!] - [\![\hat{a}_{i-1}^-, \hat{a}_{i-1}^+]\!]) \qquad i \in [2; n+m] \tag{20a}$$

$$\hat{e}_1 = \hat{a}_1^- \quad \hat{f}_1 = \hat{a}_1^+ \quad \hat{e}_i = [\![\hat{a}_{i-1}^+, \hat{a}_i^-]\!] \quad \hat{f}_i = [\![\hat{a}_i^+, \hat{a}_{i-1}^-]\!] \quad i \in [2; n+m]. \tag{20b}$$

We skip the proof of equations (20), since we give a detailed proof in the quantum case (see the theorem). Only from (18) one also derives the larger set of relation (17).

3. Description of $U_q[sl(n+1|m)]$ via deformed CAGs

In this section we define the quantum superalgebra $U_q[sl(n+1|m)]$ in terms of deformed creation and annihilation generators $a_i^\pm$, H_i, $i = 1, 2, \ldots, n+m$. The CAGs are elements from the so-called Cartan–Weyl basis of $U_q[sl(n+1|m)]$. A general procedure to construct such a basis was given in [7] (see also [29]). We follow this procedure and identify the deformed $a_1^\pm, \ldots, a_{n+m}^\pm$ generators with those elements of the Cartan–Weyl basis, which reduce to the nondeformed CAGs (16) in the limit $q \to 1$.

First we introduce $U_q[sl(n+1|m)]$ by means of its classical definition in terms of the Cartan matrix (9) and the Chevalley generators. Let $\mathbb{C}[[h]]$ be the complex algebra of formal power series in the indeterminate h, $q = e^h \in \mathbb{C}[[h]]$. $U_q[sl(n+1|m)]$ is a Hopf algebra, which is a topologically free $\mathbb{C}[[h]]$ module (complete in the h–adic topology), with (Chevalley) generators $\{h_i, e_i, f_i\}_{i\in[1;n+m]}$ subject to the Cartan–Kac relations ($\bar{q} = q^{-1}$)

$$[h_i, h_j] = 0 \tag{21a}$$

$$[h_i, e_j] = \alpha_{ij}e_j \qquad [h_i, f_j] = -\alpha_{ij}f_j \tag{21b}$$

$$[\![e_i, f_j]\!] = \delta_{ij}\frac{k_i - \bar{k}_i}{q - \bar{q}} \qquad k_i = q^{h_i} \qquad k_i^{-1} \equiv \bar{k}_i = q^{-h_i} \tag{21c}$$

the e-Serre relations (see (5))

$$[e_i, e_j] = 0 \quad \text{if} \quad |i-j| \neq 1 \qquad e_{n+1}^2 = 0 \tag{22a}$$

$$[e_i, [e_i, e_{i\pm 1}]_{\bar{q}}]_q = [e_i, [e_i, e_{i\pm 1}]_q]_{\bar{q}} = 0 \qquad i \neq n+1 \tag{22b}$$

$$\{e_{n+1}, [[e_n, e_{n+1}]_q, e_{n+2}]_{\bar{q}}\} = \{e_{n+1}, [[e_n, e_{n+1}]_{\bar{q}}, e_{n+2}]_q\} = 0 \tag{22c}$$

and the f-Serre relations, obtained from the e-Serre relations by replacing everywhere e_i with f_i:

$$[f_i, f_j] = 0 \quad \text{if} \quad |i-j| \neq 1 \qquad f_{n+1}^2 = 0 \tag{22d}$$

$$[f_i, [f_i, f_{i\pm 1}]_{\bar{q}}]_q = [f_i, [f_i, f_{i\pm 1}]_q]_{\bar{q}} = 0 \qquad i \neq n+1 \tag{22e}$$

$$\{f_{n+1}, [[f_n, f_{n+1}]_q, f_{n+2}]_{\bar{q}}\} = \{f_{n+1}, [[f_n, f_{n+1}]_{\bar{q}}, f_{n+2}]_q\} = 0. \tag{22f}$$

From (21*b*) one derives the following useful relations:

$$k_i e_j = q^{\alpha_{ij}} e_j k_i \quad k_i f_j = q^{-\alpha_{ij}} f_j k_i \quad \bar{k}_i e_j = q^{-\alpha_{ij}} e_j \bar{k}_i \quad \bar{k}_i f_j = q^{\alpha_{ij}} f_j \bar{k}_i. \tag{23}$$

We do not write the other Hopf algebra maps (Δ, ε, S) (see [7, 29]), since we will not use them. They are certainly also a part of the definition.

Remark. *We consider h as an indeterminate. All relations remain also true, if one replaces h with a number, so that q is not a root of 1. The latter corresponds to a transition from $U_q[sl(n+1|m)$ to the factor algebra $U_q[sl(n+1|m)]/h = number$.*

Following [7, 29], we introduce a normal order in the system of the positive roots $\Delta_+ = \{\varepsilon_i - \varepsilon_j | i < j \in [0; n+m]\}$ as follows:

$$\varepsilon_i - \varepsilon_j < \varepsilon_k - \varepsilon_l \quad \text{if} \quad j < l \quad \text{or if} \quad j = l \quad \text{and} \quad i < k.$$

Taking into account equations (16), we define the deformed CAGs to be Cartan–Weyl basis vectors, which are in agreement with the above normal order:

$$a_1^- = e_1 \qquad a_i^- = [[[\ldots[[e_1, e_2]_{\bar{q}_1}, e_3]_{\bar{q}_2}, \ldots]_{\bar{q}_{i-3}}, e_{i-1}]_{\bar{q}_{i-2}}, e_i]_{\bar{q}_{i-1}} = [a_{i-1}^-, e_i]_{\bar{q}_{i-1}} \tag{24a}$$

$$a_1^+ = f_1 \qquad a_i^+ = [f_i, [f_{i-1}, [\ldots, [f_3, [f_2, f_1]_{q_1}]_{q_2} \ldots]_{q_{i-3}}]_{q_{i-2}}]_{q_{i-1}} = [f_i, a_{i-1}^+]_{q_{i-1}} \tag{24b}$$

$$H_1 = h_1 \qquad H_i = h_1 + (-1)^{\theta_1} h_2 + (-1)^{\theta_2} h_3 + \cdots + (-1)^{\theta_{i-1}} h_i \tag{24c}$$

where

$$q_i = q^{1-2\theta_i} = \begin{cases} q & \text{if } i \leqslant n \\ \bar{q} & \text{if } i > n. \end{cases} \tag{25}$$

Note that equations (21)–(23) are invariant with respect to the antilinear anti-involution $(\,)^*$, defined as

$$(h)^* = -h \quad (h_i)^* = h_i \quad (e_i)^* = f_i \quad (f_i)^* = e_i \quad (ab)^* = (b)^*(a)^*. \tag{26}$$

Therefore

$$(q)^* = \bar{q} \quad (k_i)^* = \bar{k}_i \quad (a_i^\pm)^* = a_i^\mp \quad (H_i)^* = H_i. \tag{27}$$

The next proposition will be used in several intermediate computations.

Proposition 2. *The relations ($i \neq 1$)*

$$[\![e_i, a_j^-]\!]_{q_j^{\delta_{i-1,j}-\delta_{ij}}} = -q_{i-1}\delta_{i-1,j} a_i^- \tag{28a}$$

$$[\![f_i, a_j^+]\!]_{q_j^{\delta_{i-1,j}-\delta_{ij}}} = \delta_{i-1,j} a_i^+ \tag{28b}$$

$$[\![e_i, a_j^+]\!] = \delta_{ij} a_{i-1}^+ k_i^{-(-1)^{\theta_{i-1}}} \tag{28c}$$

$$[\![f_i, a_j^-]\!] = -(-1)^{\theta_{i-1,i}} \delta_{ij} k_i^{(-1)^{\theta_{i-1}}} a_{i-1}^- \tag{28d}$$

follow from (21)–(23) and the definition of the CAGs (24).

Proof. The proof is based on a case by case considerations.

(A) Consider first (28*a*).

(i) For $j \leqslant i-1$ (28*a*) is an immediate consequence of (22*a*) or of the definition (24*a*).

(ii) $j = i$.

(ii.1) For $i = 2$ (28*a*) follows directly from (22).

(ii.2) $i > 2$. Using the identity

$$\text{If} \quad [\![a,b]\!] = 0 \quad \text{then} \quad [\![[\![a,c]\!]_q, b]\!]_p = [\![a,[\![c,b]\!]_p]\!]_q \qquad p,q \in \mathbb{C}[[h]] \tag{29}$$

and the circumstance that $[e_i, a_{i-2}^-] = 0$, one obtains from (24*a*) $a_i^- = [[a_{i-2}^-, e_{i-1}]_{\bar q_{i-2}}, e_i]_{\bar q_{i-1}} = [a_{i-2}^-, [e_{i-1}, e_i]_{\bar q_{i-1}}]_{\bar q_{i-2}}$.

(ii.2a) If $i = n+1$, $[\![e_{n+1}, a_{n+1}^-]\!]_{\bar q_{n+1}} = \{e_{n+1}, [a_{n-1}^-, [e_n, e_{n+1}]_{\bar q}]_{\bar q}\}_q$.
Take into account that $[\![e_{n+1}, a_{n-1}^-]\!] = 0$ and apply the identity

$$\text{If} \quad [\![a,b]\!] = 0 \quad [\![a,[\![b,c]\!]_q]\!]_p = (-1)^{\alpha\beta}[\![b,[\![a,c]\!]_p]\!]_q$$
$$\alpha = \deg(a) \quad \beta = \deg(b). \tag{30}$$

Then $[\![e_{n+1}, a_{n+1}^-]\!]_{\bar q_{n+1}} = [a_{n-1}^-, z]_{\bar q} = 0$, since $z = \{e_{n+1}, [e_n, e_{n+1}]_{\bar q}\}_q = 0$ (from $e_{n+1}^2 = 0$).

(ii.2b) If $i \neq n+1$, then from (30) $[\![e_i, a_i^-]\!]_{\bar q_i} = [e_i, [a_{i-2}^-, [e_{i-1}, e_i]_{\bar q_{i-1}}]_{\bar q_{i-2}}]_{\bar q_i} = -\bar q_{i-1}[a_{i-2}^-, y]_{\bar q_{i-2}} = 0$, since $y = [e_i, [e_i, e_{i-1}]_{q_{i-1}}]_{\bar q_i} = 0$, according to (25) and (22*b*).

(iii) $j = i+1$.
We consider only the more difficult case, namely $i > 2$.
Set (see (24*a*)) $a_{i+1}^- = [[[a_{i-2}^-, e_{i-1}]_{\bar q_{i-2}}, e_i]_{\bar q_{i-1}}, e_{i+1}]_{\bar q_i}$. Use that $[a_{i-2}^-, e_i] = [a_{i-2}^-, e_{i+1}] = 0$ and apply twice (29): $a_{i+1}^- = [a_{i-2}^-, [[e_{i-1}, e_i]_{\bar q_{i-1}}, e_{i+1}]_{\bar q_i}]_{\bar q_{i-2}}$.

(iii.1) If $i = n+1$, $[\![e_i, a_{i+1}^-]\!] = \{e_{n+1}, a_{n+2}^-\} = \{e_{n+1}, [a_{n-1}^-, [[e_n, e_{n+1}]_{\bar q_n}, e_{n+2}]_{\bar q_{n+1}}]_{\bar q_{n-1}}\}$ (take into account that $[e_{n+1}, a_{n-1}^-] = 0$ and (30)) $= [a_{n-1}^-, \{e_{n+1}, [[e_n, e_{n+1}]_{\bar q_n}, e_{n+2}]_{\bar q_{n+1}}\}]_{\bar q_{n-1}} = 0$ according to (22*c*) and (25).

(iii.2) If $i \neq n+1$ $[\![e_i, a_{i+1}^-]\!] = [e_i, [a_{i-2}^-, [[e_{i-1}, e_i]_{\bar q_{i-1}}, e_{i+1}]_{\bar q_i}]_{\bar q_{i-2}}]$ ($[e_i, a_{i-2}^-] = 0$, use (30)) $= [a_{i-2}^-, [e_i, [[e_{i-1}, e_i]_{\bar q_{i-1}}, e_{i+1}]_{\bar q_i}]]_{\bar q_{i-2}} = [a_{i-2}^-, [e_i, [[e_{i-1}, e_i]_{\bar q_i}, e_{i+1}]_{\bar q_i}]]_{\bar q_{i-2}}$.
Apply the identity: If $\deg(a) = 0$ and $[\![b,c]\!] = 0$,

$$(x+\bar x)[a, [\![[b,a]_x, c]\!]_x] = [\![b, [a,[a,c]_x]_{\bar x}]\!]_{x^2} - [\![[a,[a,b]_x]_{\bar x}, c]\!]_{x^2}. \tag{31}$$

Then (22*b*) yields $[\![e_i, a_{i+1}^-]\!] = (\bar q_i + q_i)^{-1}[a_{i-2}^-, ([e_{i-1}, [e_i, [e_i, e_{i+1}]_{\bar q_i}]_{q_i}]_{q_i^{-2}} - [[e_i, [e_i, e_{i-1}]_{\bar q_i}]_{q_i}, e_{i+1}]_{q_i^{-2}})]_{\bar q_{i-2}} = 0$.

(iv) The case $j > i+1$ is evident. The unification of (i)–(iv) yields (28*a*).

(B) Applying the anti-involution (26) on both sides of (28*a*) one obtains (28*b*).

(C) We pass to prove (28*c*).

(i) For $i > j$ (28*c*) is an immediate consequence of (24*b*) and (21*c*).

(ii) $i = j$. $[\![e_i, a_i^+]\!] = [\![e_i, [f_i, a_{i-1}^+]_{q_{i-1}}]\!]$ (from (i) $[\![e_i, a_{i-1}^+]\!] = 0$, apply (29)) $= [\![[\![e_i, f_i]\!], a_{i-1}^+]\!]_{q_{i-1}} = [\frac{k_i - \bar k_i}{q - \bar q}, a_{i-1}^+]_{q_{i-1}} = a_{i-1}^+ k_i^{-(-1)^{\theta_{i-1}}}$.
In the last step we used the relations $k_i a_{i-1}^+ = q a_{i-1}^+ k_i$ and $\bar k_i a_{i-1}^+ = \bar q a_{i-1}^+ \bar k_i$, which follow from (24*b*) and (23).

(iii) $j = i+1$. $[\![e_i, a_{i+1}^+]\!] = [\![e_i, [f_{i+1}, [f_i, a_{i-1}^+]_{q_{i-1}}]_{q_i}]\!]$ (take into account that $[e_i, f_{i+1}] = 0$ and apply (30)) $= [\![f_{i+1}, [\![e_i, [f_i, a_{i-1}^+]_{q_{i-1}}]\!]]\!]_{q_i}$ (now $[e_i, a_{i-1}^+] = 0$, use (29)) $= [\![f_{i+1}, [\![[\![e_i, f_i]\!], a_{i-1}^+]\!]_{q_{i-1}}]\!]_{q_i} = [\![f_{i+1}, [\![\frac{k_i - \bar k_i}{q - \bar q}, a_{i-1}^+]\!]_{q_{i-1}}]\!]_{q_i} = [f_{i+1}, a_{i-1}^+ k^{-(-1)^{\theta_{i-1}}}]_{q_i}$. Using the identity $[a, bc]_x = [a,b]c + b[a,c]_x$ one has $[\![e_i, a_{i+1}^+]\!] = [f_{i+1}, a_{i-1}^+]k_i^{-(-1)^{\theta_{i-1}}} + a_{i-1}^+[f_{i+1}, k_i^{-(-1)^{\theta_{i-1}}}]_{q_i} = 0$, according to (28*b*), (23) and (25).

(iv) (28*c*) is evident for $j > i+1$. The unification of (i)–(iv) yields (28*c*).

(D) Applying the anti-involution (26) on both sides of (28c) one obtains (28d).

This completes the proof. □

Proposition 3. *The deformed CAGs (24) generate $U_q[sl(n+1|m)]$.*

Proof. Let

$$L_i = q^{H_i} \qquad \bar{L}_i \equiv L_i^{-1} = q^{-H_i}. \tag{32}$$

The proof is a consequence of the relations

$$[\![a_i^-, a_i^+]\!] = \frac{L_i - \bar{L}_i}{q - \bar{q}} \tag{33a}$$

$$[\![a_i^-, a_{i+1}^+]\!] = -(-1)^{\theta_i} L_i f_{i+1} \tag{33b}$$

$$[\![a_{i+1}^-, a_i^+]\!] = -(-1)^{\theta_i} e_{i+1} \bar{L}_i. \tag{33c}$$

We prove these equations by induction on i. For $i = 1$, (33a) holds. Let (33a) be true. Then from (28d), (30) and (33a) one has

$$\begin{aligned}[\![a_i^-, a_{i+1}^+]\!] &= [\![a_i^-, [f_{i+1}, a_i^+]_{q_i}]\!] = [\![f_{i+1}, [\![a_i^-, a_i^+]\!]]\!]_{q_i} = \frac{1}{q-\bar{q}}[f_{i+1}, L_i - \bar{L}_i]_{q_i} \\ &= \frac{1}{q-\bar{q}}[f_{i+1}, k_1 k_2^{(-1)^{\theta_1}} k_3^{(-1)^{\theta_2}} \ldots k_i^{(-1)^{\theta_{i-1}}} - \bar{k}_1 k_2^{-(-1)^{\theta_1}} k_3^{-(-1)^{\theta_2}} \ldots k_i^{-(-1)^{\theta_{i-1}}}]_{q_i}.\end{aligned}$$

Using (25) and repeatedly (23), one ends with $[\![a_i^-, a_{i+1}^+]\!] = -(-1)^{\theta_i} k_1 k_2^{(-1)^{\theta_1}} k_3^{(-1)^{\theta_2}} \ldots k_i^{(-1)^{\theta_{i-1}}} f_{i+1}$, namely with (33$b$). Similarly, one proves (33c). Therefore, if (33a) holds, then also equations (33b) and (33c) are fulfilled. Assuming this, consider $[\![a_{i+1}^-, a_{i+1}^+]\!] = [\![[a_i^-, e_{i+1}]_{\bar{q}_i}, a_{i+1}^+]\!]$. Then the identity

$$[\![[\![a,b]\!]_x, c]\!] = (-1)^{\beta\gamma}[\![[\![a,c]\!], b]\!]_x + [\![a, [\![b,c]\!]]\!]_x \qquad \beta = \deg(b) \qquad \gamma = \deg(c) \tag{34}$$

yields

$$\begin{aligned}[\![a_{i+1}^-, a_{i+1}^+]\!] &= (-1)^{\theta_{i,i+1}}[\![[\![a_i^-, a_{i+1}^+]\!], e_{i+1}]\!]_{\bar{q}_i} + [\![a_i^-, [\![e_{i+1}, a_{i+1}^+]\!]]\!]_{\bar{q}_i} \\ &= -(-1)^{\theta_{i+1}}[\![f_{i+1}, e_{i+1}]\!] k_1 k_2^{(-1)^{\theta_1}} k_3^{(-1)^{\theta_2}} \ldots k_i^{(-1)^{\theta_{i-1}}} + [\![a_i^-, a_i^+]\!] k_{i+1}^{-(-1)^{\theta_i}} \\ &= \frac{k_1 k_2^{(-1)^{\theta_1}} k_3^{(-1)^{\theta_2}} \ldots k_i^{(-1)^{\theta_{i-1}}} k_{i+1}^{(-1)^{\theta_i}} - \bar{k}_1 k_2^{-(-1)^{\theta_1}} k_3^{-(-1)^{\theta_2}} \ldots k_i^{-(-1)^{\theta_{i-1}}} k_{i+1}^{-(-1)^{\theta_i}}}{q-\bar{q}} \\ &= \frac{L_{i+1} - \bar{L}_{i+1}}{q-\bar{q}}.\end{aligned}$$

Thus, equations (33) hold for any i. From (24c) and (33) we have

$$e_1 = a_1^- \qquad e_{i+1} = -(-1)^{\theta_i}[\![a_{i+1}^-, a_i^+]\!] q^{H_i} \qquad i \in [1; n+m-1] \tag{35a}$$

$$f_1 = a_1^+ \qquad f_{i+1} = -(-1)^{\theta_i}\bar{q}^{H_i}[a_i^-, a_{i+1}^+]\!] \qquad i \in [1; n+m-1] \tag{35b}$$

$$h_1 = H_1 \qquad h_i = (-1)^{\theta_{i-1}}(H_i - H_{i-1}) \qquad i \in [2; n+m] \tag{35c}$$

which completes the proof. □

We proceed to state our main result.

Theorem. *$U_q[sl(n+1|m)]$ is an unital associative algebra, which is topologically free $\mathbb{C}[[h]]$ module, with generators $\{H_i, a_i^{\pm}\}_{i\in[1;n+m]}$ and relations*

$$[H_i, H_j] = 0 \qquad (36a)$$

$$[H_i, a_j^{\pm}] = \mp(1 + (-1)^{\theta_i}\delta_{ij})a_j^{\pm} \qquad (36b)$$

$$[\![a_i^-, a_i^+]\!] = \frac{L_i - \bar{L}_i}{q - \bar{q}} \qquad (36c)$$

$$[\![[\![a_i^{\eta}, a_{i+\xi}^{-\eta}]\!], a_k^{\eta}]\!]_{q^{\xi(1+(-1)^{\theta_i}\delta_{ik})}} = \delta_{k,i+\xi}L_k^{-\xi\eta}a_i^{\eta} \qquad \xi,\eta = \pm \quad or \quad \pm 1 \qquad (36d)$$

$$[\![a_1^{\xi}, a_2^{\xi}]\!]_q = 0 \qquad [\![a_1^{\xi}, a_1^{\xi}]\!] = 0 \qquad \xi = \pm. \qquad (36e)$$

Proof. As a first step one has to show that equations (36) hold. Most of the results for this part of the proof are already obtained. Equation (36a) is evident. Equation (36b) follows from the relation $\sum_{p=1}^{i}\sum_{q=1}^{j}(-1)^{\theta_{p-1}}\alpha_{pq} = 1 + (-1)^{\theta_i}\delta_{ij}$, the definitions of $a_i^{\pm}$ and H_i (see (24)) and the relations (21b). From (36b) one also derives

$$L_i a_j^{\pm} = q^{\mp(1+(-1)^{\theta_i}\delta_{ij})}a_j^{\pm}L_i. \qquad (37)$$

Equation (36c) is the same as (33a). The derivation of all triple relations (36d) is relatively long, but simple. It is based on a case by case consideration. To this end one replaces e_i and f_i in (28) with the right-hand sides of (35a), (35b). The nontrivial part is to put all cases in the compact form (36d). If $n \neq 0$, $[\![a_1^{\xi}, a_1^{\xi}]\!] = [a_1^{\xi}, a_1^{\xi}] = 0$. The first relations in (36e) reduce to the triple Serre relations (22b), (22e). If $n = 0$, equations (36e) hold because $e_1^2 = 0$ and $f_1^2 = 0$.

It remains to prove as a second step that any other relation in $U_q[sl(n+1|m)]$ follows from equations (36). To this end it suffices to show that all Cartan–Kac relations (21) and the Serre relations (22) follow from (36).

(A) The Cartan–Kac relations (21a) follow in an evident way from (35c) and (36a).

(B) Equations (21b) are easily derived from (35) and (36b).

(C) The proof of (21c) is not trivial. It is based on the identity ($\alpha = \deg(a)$, $\beta = \deg(b)$): If $x = zs$, $y = zr$, $t = zsr$; $x, y, z, r, s, t \in \mathbb{C}[[h]]$, then

$$[\![a, [\![b, c]\!]_x]\!]_y = [\![[\![a, b]\!]_z, c]\!]_t + z(-1)^{\alpha\beta}[\![b, [\![a, c]\!]_r]\!]_s. \qquad (38)$$

(i) The case $i, j \in [2; n+m]$. From (35) and (37) one derives

$$[\![e_i, f_j]\!] = (-1)^{\theta_{i-1,j-1}}q^{(-1)^{\theta_{i-1}-(-1)^{\theta_{j-1}}\delta_{ij}+(-1)^{\theta_{j-1}}\delta_{i,j-1}}}L_{i-1}\bar{L}_{j-1}$$
$$\times[\![[\![a_i^-, a_{i-1}^+]\!], [\![a_{j-1}^-, a_j^+]\!]]\!]_{q^{(-1)^{\theta_{i-1}}\delta_{i-1,j}-(-1)^{\theta_{j-1}}\delta_{i,j-1}}} \qquad (39)$$

(i.1) $i = j$. $[\![e_i, f_i]\!] = [\![[\![a_i^-, a_{i-1}^+]\!], [\![a_{i-1}^-, a_i^+]\!]]\!]$ (apply (38) with $a = [\![a_{i-1}^+, a_i^-]\!]$, $b = a_i^+$, $c = a_{i-1}^-$ and $x = y = 1$, $z = q$, $r = s = t = \bar{q}$ and use (36) and (37)) $= \frac{(-1)^{\theta_{i-1}}}{q-\bar{q}}(k_i^{(-1)^{\theta_{i-1}}} - k_i^{-(-1)^{\theta_{i-1}}}) = \frac{k_i-\bar{k}_i}{q-\bar{q}}$.

(i.2) Let $|i-j| > 1$. $[\![e_i, f_j]\!] = (-1)^{\theta_{ij}}q^{(-1)^{\theta_{i-1}}}L_{i-1}\bar{L}_{j-1}[\![[\![a_{i-1}^+, a_i^-]\!], [\![a_j^+, a_{j-1}^-]\!]]\!]$ (use (38) with $a = [\![a_{i-1}^+, a_i^-]\!]$, $b = a_j^+$, $c = a_{j-1}^-$, $x = y = 1$, $t = s = r = \bar{z} = \bar{q}$ and (36d)) $= 0$.

(i.3) The cases $j = i-1$ and $j = i+1$ are proved in a similar way as (i.2).

(ii) The verification of (21c) for $i = j = 1$; $i = 1$, $j > 1$ and $i > 1$, $j = 1$ is simple.

(D) We pass to prove (22a), namely that $[\![e_i, e_j]\!] = 0$ if $|i-j| \neq 1$.

(i) The cases with $i = 1$ and $j \in [3; n+m]$ follow directly from (37) and (36d).

(ii) $i \neq j \in [2; n+m]$. From (35*a*) and (37) $[e_i, e_j] = (-1)^{\theta_{ij}}[[\![a^+_{i-1}, a^-_i]\!], [\![a^+_{j-1}, a^-_j]\!]]L_{i-1}L_{j-1}$ (use (38) with $a = [\![a^+_{i-1}, a^-_i]\!]$, $b = a^+_{j-1}$, $c = a^-_j$, $x = y = 1$, $t = r = s = \bar{z} = \bar{q}$ and (36*d*)) = 0.

(iii) If $i = j \neq n+1$, $[\![e_i, e_i]\!] = [e_i, e_i] = 0$.

(iv) Consider $e^2_{n+1} = \frac{1}{2}\{e_{n+1}, e_{n+1}\} \equiv \frac{1}{2}[\![e_{n+1}, e_{n+1}]\!]$.

(iv.1) The case with $n+1 = 1$ is evident: $\{e_1, e_1\} = \{a^-_1, a^-_1\} = 0$, see (36*e*).

(iv.2) $n+1 \neq 1$. Use (35*a*): $e^2_{n+1} \sim \{e_{n+1}, e_{n+1}\}_{q^2} = \bar{q}[\![[\![a^-_{n+1}, a^+_n]\!], [\![a^-_{n+1}, a^+_n]\!]]\!]_{q^2}L^2_n$ (from (38) with $a = [\![a^-_{n+1}, a^+_n]\!]$, $b = a^-_{n+1}$, $c = a^+_n$, $x = s = z = 1$, $y = r = t = q^2$) = 0. Hence the Serre relations (22*a*) follow from (36).

(E) We prove the triple Serre relation $[e_i, [e_i, e_{i+1}]_{\bar{q}}]_q = [e_i, [e_i, e_{i+1}]_q]_{\bar{q}} = 0, \quad i \neq n+1$.

(i) The case $i = 1$ is evident: $[e_1, [e_1, e_2]_{\bar{q}}]_q = [a^-_1, a^-_2]_q = 0$.

(ii) $i \in [2; n]$. From (35*a*) and (37) $[e_i, e_{i+1}]_{\bar{q}} = [[a^-_i, a^+_{i-1}]L_{i-1}, [a^-_{i+1}, a^+_i]L_i]_{\bar{q}} = [[a^-_i, a^+_{i-1}], [a^-_{i+1}, a^+_i]]L_iL_{i-1}$ (apply (38) with $a = [a^-_i, a^+_{i-1}]$, $b = a^-_{i+1}$, $c = a^+_i$, $x = y = 1$, $z = \bar{q}$, $r = s = t = q$) = $[[[a^-_i, a^+_{i-1}], a^-_{i+1}]_{\bar{q}}, a^+_i]_qL_iL_{i-1} + \bar{q}[a^-_{i+1}, [[a^-_i, a^+_{i-1}], a^+_i]_q]_qL_iL_{i-1}$ (use (36*d*) and (37)) = $-\bar{q}[a^-_{i+1}, \bar{L}_ia^+_{i-1}]_qL_iL_{i-1} = -[a^-_{i+1}, a^+_{i-1}]L_{i-1}$. Therefore $[e_i, [e_i, e_{i+1}]_{\bar{q}}]_q = [[a^-_i, a^+_{i-1}]L_{i-1}, [a^-_{i+1}, a^+_{i-1}]L_{i-1}]_q = \bar{q}[[a^-_i, a^+_{i-1}], [a^-_{i+1}, a^+_{i-1}]]_qL^2_{i-1}$ (from (38) with $a = [a^-_i, a^+_{i-1}]$, $b = a^-_{i+1}$, $c = a^+_{i-1}$, $x = 1$, $y = s = q$, $z = \bar{q}$, $r = t = q^2$) = $\bar{q}([[[a^-_i, a^+_{i-1}], a^-_{i+1}]_{\bar{q}}, a^+_{i-1}]_{q^2}]_q + \bar{q}[a^-_{i+1}, [[a^-_i, a^+_{i-1}], a^+_{i-1}]_{q^2}]_q)L^2_{i-1} = 0$, according to (36*d*).

(iii) $i \in [n+2; n+m]$. The proof is similar to (ii).

The other triple e-Serre relation $[e_i, [e_i, e_{i-1}]_{\bar{q}}]_q = [e_i, [e_i, e_{i-1}]_q]_{\bar{q}} = 0$ is proved in a similar way.

(F) Contrary to the proof of proposition 2, equations (28) are an easy consequence of (38). Note also that from equations (28) one derives equations (24*a*), (24*b*). We use them in order to prove the additional Serre relation (22*c*).

Using (24*a*), write $a^-_{n+2} = [[[a^-_{n-1}, e_n]_{\bar{q}}, e_{n+1}]_{\bar{q}}, e_{n+1}]_q$. From (28*a*) $\{e_{n+1}, a^-_{n+2}\} = 0$. Therefore $0 = \{e_{n+1}, a^-_{n+2}\} = \{e_{n+1}, [[[a^-_{n-1}, e_n]_{\bar{q}}, e_{n+1}]_{\bar{q}}, e_{n+1}]_q\}$. Since $[e_{n+1}, a^-_{n-1}] = [e_{n+2}, a^-_{n-1}] = 0$ (see (28*a*)), applying twice (29) and (30), one obtains $\{e_{n+1}, a^-_{n+2}\} = [a^-_{n-1}, y]_{\bar{q}} = 0$, where

$$y = \{e_{n+1}, [[e_n, e_{n+1}]_{\bar{q}}, e_{n+2}]_q\}. \tag{40}$$

Therefore $[y, a^-_{n-1}]_q = 0$. From (24*b*), (21*c*) and (40) it follows also that $[y, a^+_{n-1}] = 0$. Applying (29) we have $0 = [[y, a^-_{n-1}]_q, a^+_{n-1}] = [y, [a^-_{n-1}, a^+_{n-1}]]_q$ (use (36*c*), (24c), (21*b*)) = $(q - \bar{q})^{-1}[y, L_{n-1} - \bar{L}_{n-1}]_q = qy\bar{L}_{n-1}$. Hence, $y = 0$, i.e. the additional e-Serre relation (22*c*) holds.

(G) In a similar way one derives the f-Serre relations (22*d*)–(22*f*). Another way to prove them is to apply the $*$-operation (26) on both sides of the e-Serre relations.

This completes the proof of the theorem. □

4. Discussions and further outlook

In this paper we enlarge the list of the quantum superalgebras, which admit a description via deformed creation and annihilation generators [8–13], adding to it all quantum superalgebras $U_q[sl(n+1|m)]$. The possibility for such a description is not unexpected. We have generalized the results for $U_q[sl(n+1)]$ [13] to the superalgebra case. This generalization is, however, we wish to point out, neither evident nor straightforward. The 'super' structure is richer, with more relations ($e_{n+1}^2 = f_{n+1}^2 = 0$, additional Serre relations (22c), (22f)) and, as a result, with several features which do not appear in the Lie algebra cases (the simple root systems are not related by transformations from the Weyl group, one and the same superalgebra admits several Dynkin diagrams, etc). All these peculiarities, especially in the deformed case, which we have mainly in mind here and below, make the computations nontrivial and technically much more involved.

In the introduction we said a few words to justify the name *creation and annihilation generators*. Another reason for this name stems from the observation that, using the CAGs, one can construct Fock spaces in a much similar way as in the parastatistics quantum field theory (postulating the existence of a vacuum, which is annihilated by all a_i^- operators and introducing an order of the statistics [16]; for more details on parastatistics see, for instance, [32]). Then the Fock spaces are generated by the creation operators, acting on the vacuum. Moreover a_i^+, acting on a state with fixed number of 'particles' (elementary excitation) of species i, increases them by one, whereas a_i^- diminishes them by one. The advantage of this property for the physical applications and interpretation is evident. Consider, for instance, a 'free' Hamiltonian

$$H = \sum_{i=1}^{n+m} \varepsilon_i H_i \qquad \text{such that} \quad \sum_{i=1}^{n+m} (-1)^{\theta_i} \varepsilon_i = 0 \tag{41}$$

which in the nondeformed case takes the usual form

$$H = \sum_{i=1}^{n+m} \varepsilon_i [\![a_i^+, a_i^-]\!]. \tag{42}$$

Then

$$[H, a_i^{\pm}] = \pm \varepsilon_i a_i^{\pm} \tag{43}$$

i.e. a_i^+ (resp. a_i^-) can be interpreted as an operator creating (resp. annihilating) a 'particle' of species i with energy ε_i. Our *physical conjecture* is that the Fock representations of the deformed CAGs will lead to new solutions for the microscopic g-ons statistics in the sense of Karabali and Nair [33], which is a particular realization of the exclusion statistics of Haldane [27].

The Fock representations may, however, also be of interest from another, more mathematical point of view. So far the finite-dimensional irreducible representations of the LSs from the class A were explicitly constructed only for $sl(n|1)$ [34]. Any such representation can be deformed to a representation of $U_q[sl(n|1)]$ [35]. The representation theory of $sl(n|m), n, m = 1, 2, \ldots$ and hence of the corresponding deformed algebras is, however, far from being complete, if both $n \neq 1$ and $m \neq 1$. In [36] the so-called essentially typical representations of $sl(n|m)$ were described. The results were also generalized to the quantum case [37]. Our *mathematical conjecture* is that the Fock representations are beyond the class of the deformed essentially typical representation [36], thus yielding new representations of $U_q[sl(n + 1|m)]$.

In order to verify the above conjectures one would need to construct the Fock representations explicitly, i.e. to introduce a basis and to write down the transformations of the basis under the action of the generators. As a first step one has to determine the quantum analogue of the triple relations (17). This is a nontrivial problem. It actually means that one has to write down the supercommutation relations between all Cartan–Weyl generators, expressed via the CAGs. The latter is a necessary condition for the application of the Poincare–Birghoff–Witt theorem, when computing the action of the generators on the Fock basis vectors. We return to this problem elsewhere. Here we mention only one, but important additional relation: from (17) one derives that the creation (resp. annihilation) generators q-supercommute,

$$[\![a_i^\xi, a_j^\xi]\!]_{q'} = 0 \qquad q' = q \quad \text{or} \quad \bar{q} \qquad i, j \in [1; n+m] \quad \xi = \pm. \tag{44}$$

This makes evident the basis (or at least one possible basis) in a given Fock space, since any product of only creation generators can always be ordered. Note that similar property does not hold for para-Bose (or para-Fermi) creation operators. This is the reason why (even in the nondeformed case) the matrix elements of the para-operators still remains unknown for an arbitrary order of the parastatistics: the Fock space basis cannot be represented as ordered products of only para-Bose (or para-Fermi) creation operators acting on the vacuum (the linear span of such vectors is not invariant under the action of the para-operators).

Finally let us mention that we do not have simple relations for the action of the other Hopf algebra operations (Δ, ε, S) on the CAGs, although it is clear how to write them down, using equations (16) and the circumstance that the comultiplication Δ and the co-unity ε are morphisms, whereas the antipode S is an antimorphism. In this respect the picture is much the same as discussed in [13]. Luckily, the (Δ, ε, S)-operations are not necessary for computing the transformations of the Fock modules (but they are certainly very important when considering tensor products of representation spaces).

Acknowledgment

We are thankful to Professor Randjbar-Daemi for the kind invitation to visit the High Energy Section of the Abdus Salam International Centre for Theoretical Physics.

References

[1] Drinfeld V G 1985 *DAN SSSR* **283** 1060
Drinfeld V G 1985 *Sov. Math. Dokl.* **32** 254
[2] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
[3] Kulish P P 1988 *Zapiski Nauch. Semin. LOMI* **169** 95
[4] Kulish P P and Reshetikhin N Yu 1989 *Lett. Math. Phys.* **18** 143
[5] Chaichian M and Kulish P P 1990 *Phys. Lett.* B **234** 72
[6] Bracken A J, Gould M D and Zhang R B 1990 *Mod. Phys. Lett.* A **5** 831
[7] Tolstoy V N 1990 *Lecture Notes in Physics* **370** (Berlin: Springer) p 118
[8] Palev T D 1993 *J. Phys. A: Math. Gen.* **26** L1111
(Palev T D 1993 *Preprint* hep-th/9306016)
[9] Hadjiivanov L K 1993 *J. Math. Phys.* **34** 5476
[10] Palev T D and Van der Jeugt J 1995 *J. Phys. A: Math. Gen.* **28** 2605
(Palev T D and Van der Jeugt J 1995 *Preprint* q-alg/9501020)
[11] Palev T D 1994 *Lett. Math. Phys.* **31** 151
(Palev T D 1993 *Preprint* het-th/9311163)
[12] Palev T D 1998 *Commun. Math. Phys.* **196** 429
(Palev T D 1997 *Preprint* q-alg/9709003)
[13] Palev T D and Parashar P 1998 *Lett. Math. Phys.* **43** 7
(Palev T D and Parashar P 1996 *Preprint* q-alg/9608024)
[14] Palev T D 1980 *J. Math. Phys.* **21** 1293
[15] Palev T D 1982 *J. Math. Phys.* **23** 1100
[16] Green H S 1953 *Phys. Rev.* **90** 270
[17] Kac V G 1979 *Lecture Notes in Math.* **676** (Berlin: Springer) p 597
[18] Palev T D 1979 *Czech. J. Phys.* B **29** 91
[19] Palev T D 1976 Lie algebraical aspects of the quantum statistics *Thesis* Institute for Nuclear Research and Nuclear Energy, Sofia
[20] Palev T D 1977 Lie algebraical aspects of the quantum statistics. Unitary quantization (A-quantization) *Preprint* JINR E17-10550 and hep-th/9705032

Palev T D 1980 *Rep. Math. Phys.* **18** 117
Palev T D 1980 *Rep. Math. Phys.* **18** 129
[21] Palev T D 1978 A-superquantization *Communications JINR* E2-11942
[22] Palev T D 1982 *J. Math. Phys.* **23** 1778
Palev T D 1982 *Czech. J. Phys.* B **32** 680
[23] Okubo S 1994 *J. Math. Phys.* **35** 2785
[24] Van der Jeugt J 1996 *New Trends in Quantum Field Theory* (Sofia: Heron Press)
[25] Meljanac S, Milekovic M and Stojic M 1998 *Mod. Phys. Lett.* A **13** 995
(Meljanac S, Milekovic M and Stojic M 1997 *Preprint* q-alg/9712017)
[26] Palev T D and Stoilova N I 1997 *J. Math. Phys.* **38** 2506
(Palev T D and Stoilova N I 1996 *Preprint* hep-th/9606011)
[27] Haldane F D M 1991 *Phys. Rev. Lett.* **67** 937
[28] Palev T D 1992 *Rev. Mod. Phys.* **31** 241
[29] Khoroshkin S M and Tolstoy V N 1991 *Commun. Math. Phys.* **141** 599
[30] Floreanini R, Leites D A and Vinet L 1991 *Lett. Math. Phys.* **23** 127
[31] Scheunert M 1992 *Lett. Math. Phys.* **24** 173
[32] Ohnuki Y and Kamefuchi S 1982 *Quantum Field Theory and Parastatistics* (Berlin: Springer)
[33] Karabali D and Nair V P 1995 *Nucl. Phys.* B **438** 551
[34] Palev T D 1989 *J. Math. Phys.* **30** 1433
[35] Palev T D and Tolstoy V N 1991 *Commun. Math. Phys.* **141** 549
[36] Palev T D 1989 *Funkt. Anal. Prilozh.* **23** 69 (in Russian) (Engl. Transl. 1989 *Funct. Anal. Appl.* **23** 141)
[37] Palev T D, Stoilova N I and Van der Jeugt J 1994 *Commun. Math. Phys.* **166** 367